Artistic Islamic astrolabes in the light of modern geometric concepts

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2024

This talk.

1. Some beautiful astrolabe pictures, from the Lahore school
2. What is an astrolabe?
3. Modern geometry: orthogonal systems of coaxal circles.
4. So what?

1. Islamic astrolabes: pictures

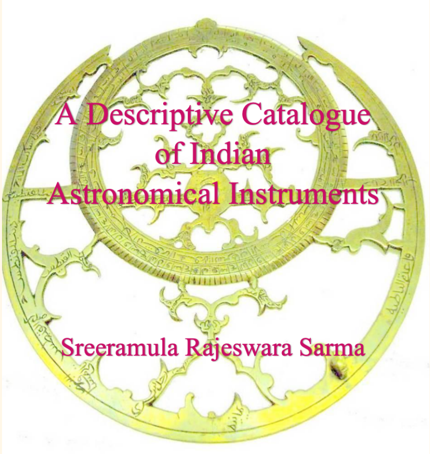
We will quote from the magnificent ca. 4500 pages work:

S.R. Sarma, *A Descriptive Catalogue of Indian Astronomical Instruments*

Available on the internet for free (Ca. 400 MB), see

www.srsarma.in

ca. 1500 pages on the Lahore astronomical school.



Extremely beautiful astrolabes were produced in Lahore.

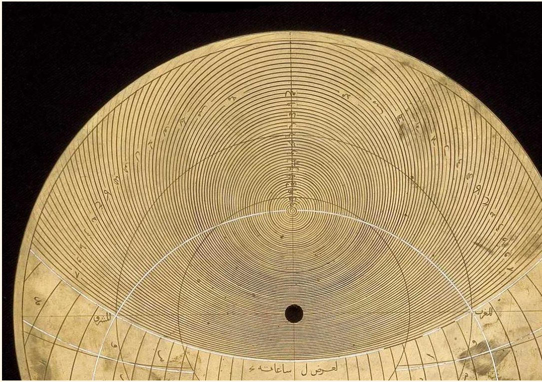
Astrolabe by Ḍiya al-Din,
1054 H (1653-54 AD)
diameter 102 mm!
auctioned in London, 2009
now in private collection
Sarma no. A071, p. 802.

An astrolabe by the founder of the Lahore astrolabe school

Astrolabe by Allahdad, ca. 1570 AD
diameter 255 mm
Oxford, Museum for the History of Science
no. 47376
Sarma no. A002, p. 105.

These astrolabes were precision instruments! We will be looking at this pattern of nearly concentric circles

Astrolabe by Allahdad, ca. 1570 AD
Plate for latitude 30^o
diameter 255 mm
Oxford, Museum for the History of Science
no. 47376 (tympan 1b)



Simpler Lahore astrolabe by Ḍiya al-Dīn (1047 H./1637-8 CE)

Star map can rotate on top of a plate with horizon coordinate system.

Adler Planetarium, Chicago (USA), no. A-86. diameter 117 mm Sarma no. A60 p. 711.

Star map with thick circle (path of sun); little hole in the middle is north pole in the sky.

Path of the sun is ecliptic, divided into 12 signs

Ḍiya al-Dīn (1047 H./1637-8 CE) Adler Planetarium, Chicago (USA), no. A-86. diameter 117 mm

If the star map rotates over the plate, the astrolabe imitates what happens in nature.

Problems which the astrolabe could solve

Find the time of day from the height of the sun in the sky?

Find the time at night from the height of a bright star?

Find the directions (North, East, etc.) during the day?

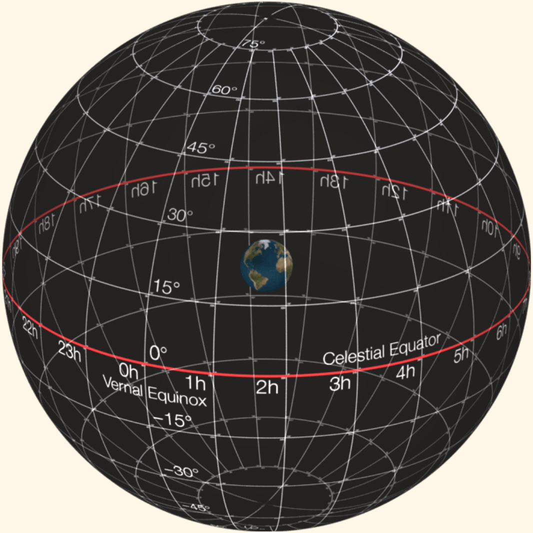
Find prayer times and qibla?

WIthout clocks, watches, compasses.

2. What is an astrolabe? Now we become more technical. First we look at the celestial sphere.

A sphere with a very large radius, and the observer at the centre,
so large that the radius of the earth can be ignored,

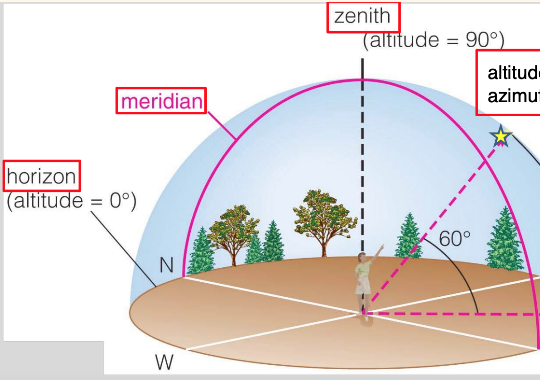
we can define coordinate systems on the celestial sphere
example: equatorial system



2. Horizon coordinate system on the celestial sphere.

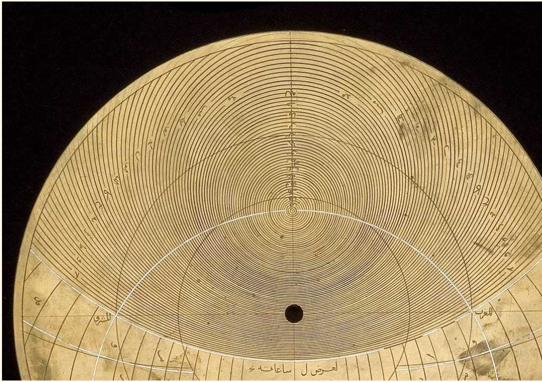
The coordinates are called altitude (above the horizon of the observer, maximum 90 at zenith - Arabic samt al-ra's) and azimuth (direction: Arabic al-samt, plural al-sumūt)

Meridian circle is an azimuth circle through north and south points on the horizon.



2. What is an astrolabe?

Plate is mapping of coordinate system on the celestial sphere.
Black hole: celestial north pole (close to Pole Star)
Silvery nearly horizontal circle: horizon.
One more silvery circle, from left to right: intersects the horizon on the left side at the East point, goes through the zenith (intersection with the vertical line), intersects the horizon on the right side at the West point



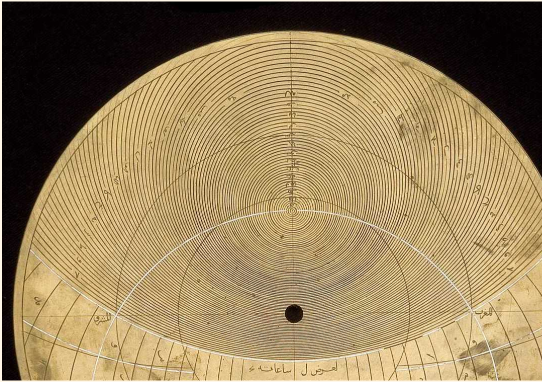
2. What is an astrolabe?

vertical line: meridian, from North point on the horizon, upwards to zenith and then further upwards to South point on the horizon (but this is outside the plate and therefore not visible).

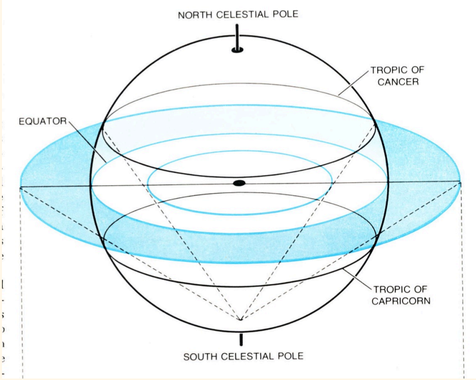
Point inside all the non-silvery circles: zenith (point above your head)

Non-silvery circles above the horizon: altitude circles, called almucantar, plural almucantarat

azimuthal circles are drawn below the horizon to prevent an unclear grid.



Principle of the astrolabe: stereographic projection from the (invisible) south pole maps the celestial sphere (left) on the plane of the equator (right). Here you see the map of the tropics of Cancer and Capricorn.



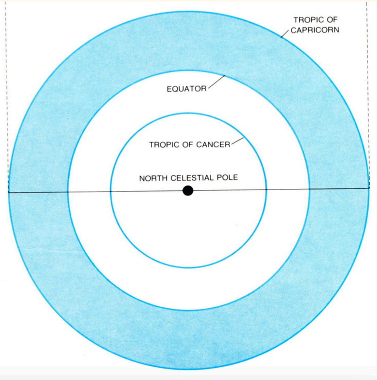
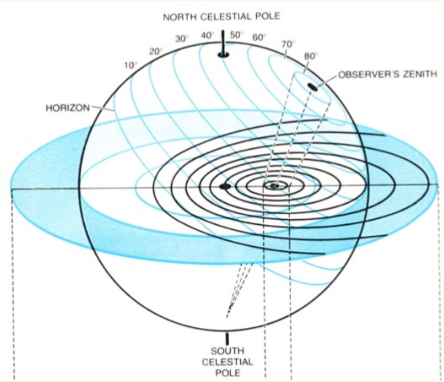


Figure credit: J. D. North, The Astrolabe, 1974.

Horizon and almucantar curves (circles of equal altitude) on astrolabe plate are also obtained by stereographic projection from south pole of the celestial sphere.



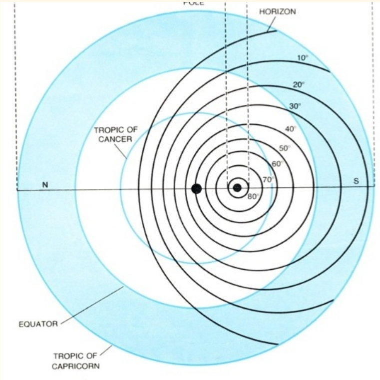
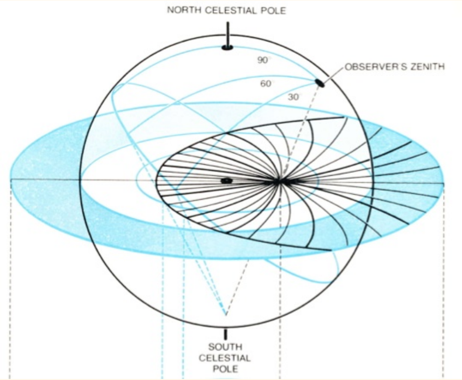


Figure credit: J. D. North, The Astrolabe, 1974.

Azimuthal curves (equal direction) are also obtained by stereographic projection from the south pole - these curves are an innovation made in the early Islamic tradition



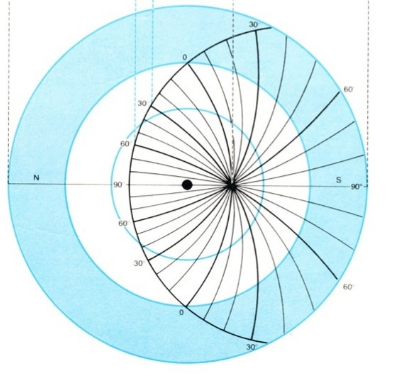
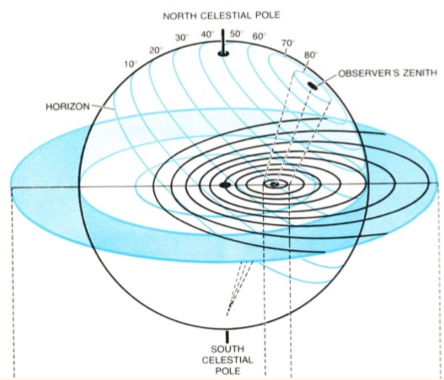
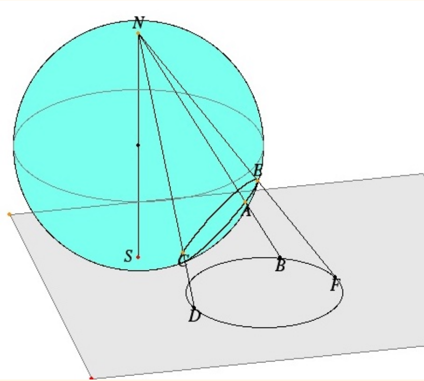


Figure credit: J. D. North, The Astrolabe, 1974.

Note that stereographic projection maps circles on the sphere to circles (and sometimes straight lines) in the plane. This is why the astrolabe could be constructed in practice.

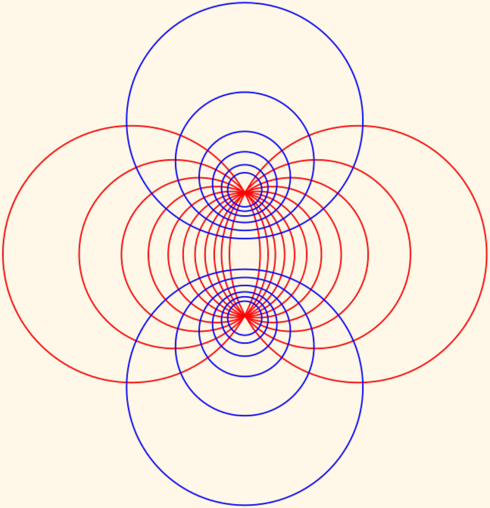




3.We will now look at (orthogonal) systems of coaxal circles

Geometric concept first defined in the 19th century.

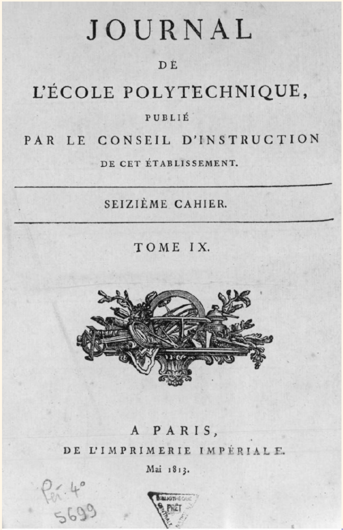
The almucantars and azimuthal circles of an astrolabe will turn out to be orthogonal systems of coaxal circles.



3. Orthogonal systems of coaxal circles were first defined in 1813

Article: "On the general means to graphically construct a circle determined by three conditions, and a sphere determined by four conditions".

Author: L. Gaultier, Professor of descriptive geometry, on the *Conservatoire des Arts et Métiers* in Paris.


JOURNAL

DE

L'ÉCOLE POLYTECHNIQUE,

PUBLIÉ

PAR LE CONSEIL D'INSTRUCTION

DE CET ÉTABLISSEMENT.

SEIZIÈME CAHIER.

TOME IX.

A PARIS,

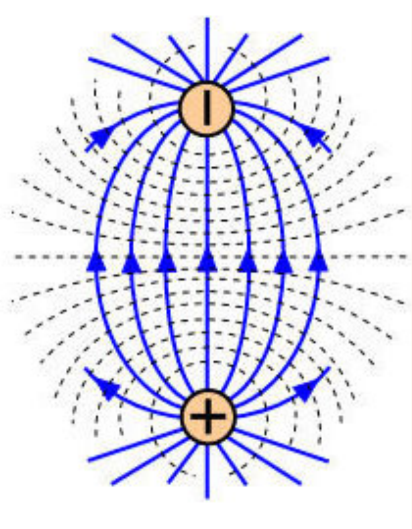
DE L'IMPRIMERIE IMPÉRIALE.

Mai 1813.

3. Orthogonal systems of coaxal circles have important applications today.

Example: electric field

(field lines and equipotential lines) generated by positive and negative electric charge.

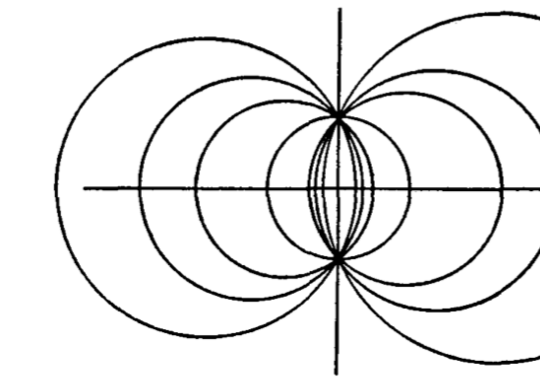


3. Simplified definition - enough for our purpose today. We will define two systems of coaxal circles.

First system: all circles through two fixed points.

"Radical axis" of the system is the (vertical) straight line through the two fixed points. Coaxal means: having the same radical axis.
(we will not explain the meaning of "radical axis")

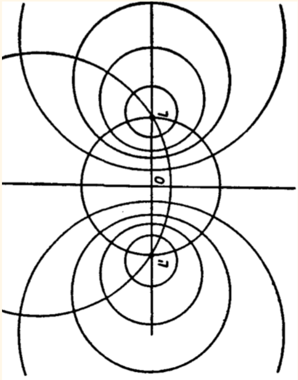
The centres of the circles are all on the horizontal line.



3. Simplified definition - enough for our purpose today. We will define two types of systems of coaxal circles.

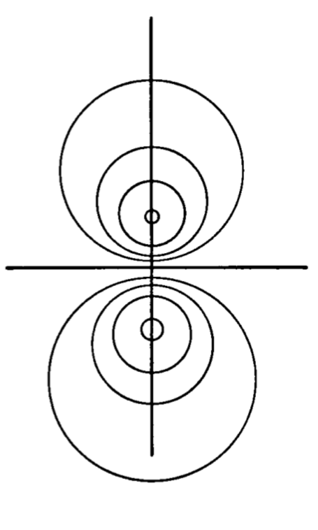
Second system: all circles "around" the two fixed points, which intersect the circles in the first system orthogonally (that is, perpendicularly)

The centres of these circles are on the vertical line in the figure.
The horizontal line in the figure is the "radical axis" of the system.



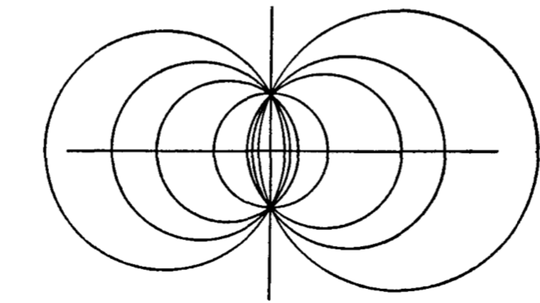
3. Second system without the first system.

(These are so-called circles of Apollonius with respect to the fixed points C_1, C_2: each circle consists of all points P such that
$PC_1 : PC_2 = \lambda$
where $\lambda \neq 1$ is a constant value)



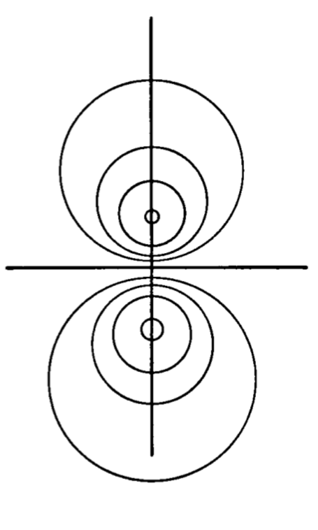
3. First system of coaxal circles in coordinates.

Let $c > 0$ be constant, and consider the circles $x^2 + y^2 - 2ax - c = 0$ for each real number a. All these circles pass through the two fixed points $(x, y) = (0, \pm\sqrt{c})$



3. Second system of coaxal circles in coordinates

Let $c > 0$ be the same constant, and consider the circles $x^2 + y^2 - 2by + c = 0$ for each real number b.
The circle has centre $(0, b)$ and radius $\sqrt{b^2 - c}$, therefore $|b| \geq \sqrt{c}$. The two point circles $(0, \pm\sqrt{c})$ are the fixed points of the system (usually called the two "limiting points").



3. These two systems are orthogonal: the circles intersect orthogonally

Let $c > 0$, the systems are
(1) $x^2 + y^2 - 2ax - c = 0$
(2) $x^2 + y^2 - 2by + c = 0$
for all real numbers a, b.
$(0, \pm\sqrt{c})$ are the fixed points.

As you can see in the figure, the circles of the first system intersect the circles in the second system orthogonally.

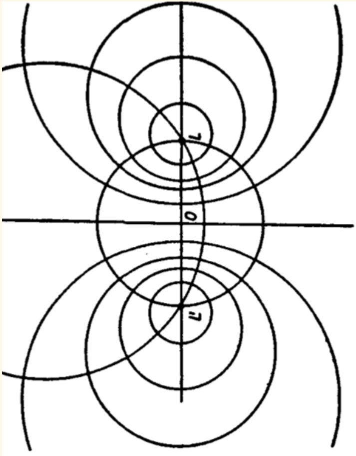


image credit: Daniel Pedoe, *Circles: a mathematical view*, Fig 22.

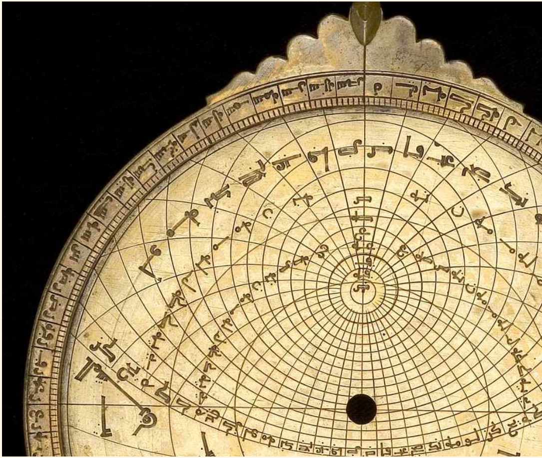
3. Analytic proof of orthogonality (for you to read)

Let $c > 0$, the systems are (1) $x^2 + y^2 - 2ax - c = 0$ and (2) $x^2 + y^2 - 2by + c = 0$ for all real numbers a, b.
Suppose that a circles in system (1) intersects a circle in system (2) at point $P = (x, y)$.
The centre of circle (1) is at $A = (a, 0)$ and the centre of circle (2) is at $B = (0, b)$.
Consider the vectors (radii) $PA = (x - a, y)$ and $PB = (x, y - b)$.
The vectors are perpendicular if their inner product $(x - a).x + y(y - b)$ is zero. (3).
This can be shown by eliminating c from (1) and (2): then
$x^2 + y^2 - 2ax = -x^2 - y^2 + 2by$,
therefore $2x^2 + 2y^2 - 2ax - 2by = 0$, so (3) follows.
Since the radii ending at P are perpendicular, the tangents to the circles at P are also perpendicular.
Therefore the circles intersect orthogonally.

The astrolabe and the first system of coaxal circles

the azimuthal circles are a first system of coaxal circles.
They are all circles passing through two fixed points: the zenith . . .).

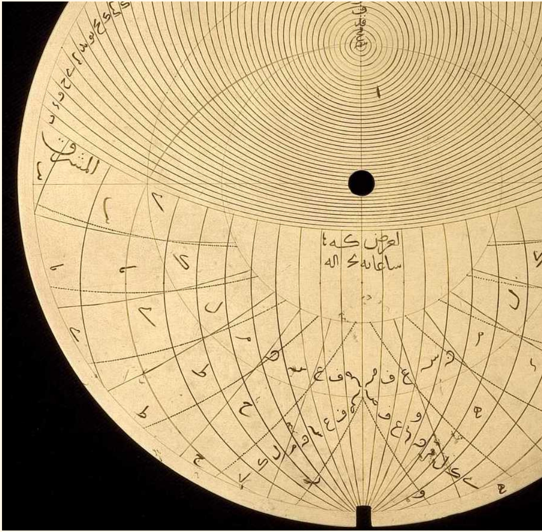
Astrolabe made in Toledo 460 AH (1068 AD), Oxford History of Science Museum no. 55331.
Plate for latitude $28^o 20'$.



The astrolabe and the first system of coaxal circles

the azimuthal circles are a first system of coaxal circles.
They are all circles passing through two fixed points: the zenith and the nadir (Arabic: naẓīr).

Astrolabe attributed to Allahdad, Lahore, ca.1570, plate for 25^o
Sarma A003, p. 136, Oxford History of Science Museum no. 34611.

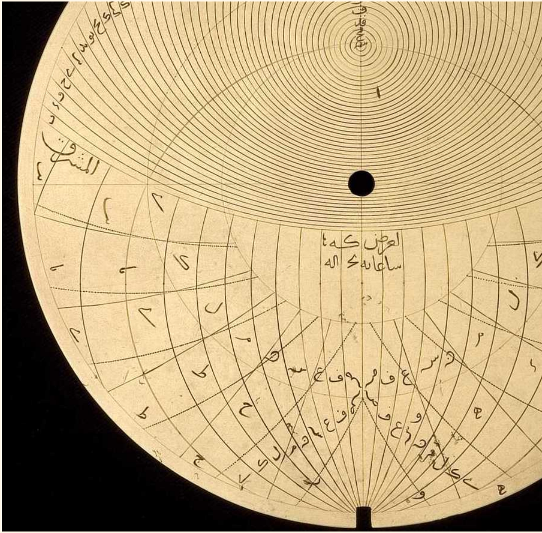


Formulas:

the azimuthal circles are a first system of coaxal circles of type:
$x^2 + y^2 - 2ax - c = 0$.

If ϕ is the geographical latitude of the locality for which the plate was made, ζ the azimuth (reckoned from the north), and R the radius of the equator on the astrolabe, we have
$c = R\frac{1}{\cos^2\phi}$,
$a = R\frac{1}{\tan\zeta\cos^2\phi}$,
the centre of the astrolabe (pole) is at $(0, R\tan\phi)$.

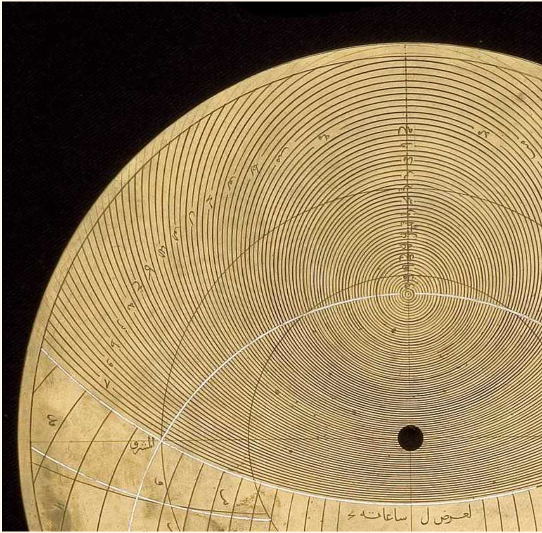
Sarma A003, Oxford no. 34611, Lahore ca. 1570.



The astrolabe and the second system of coaxal circles

the almucantar (altitude) circles are a second system of coaxal circles.
The fixed points are the zenith (visible) and the nadir (invisible). The radical axis is below the horizon and therefore also invisible.

Sarma A003, Oxford no. 34611, Lahore ca. 1570, plate for 30^o.



Formulas:

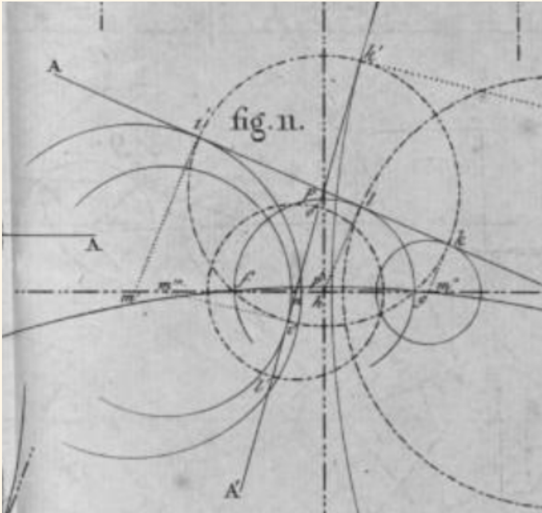
the almucantar circles are a second system of coaxal circles:
$x^2 + y^2 - 2by + c = 0$

If ϕ is the geographical latitude of the locality for which the plate was made,
z the zenith distance of the altitude circle, and R the radius of the equator on the astrolabe,
then the centre of the astrolabe (pole) is at $(0, R\tan\phi)$ and
$c = R\frac{1}{\cos^2\phi}$, as above.
We have $b = R\tan\phi + \frac{R}{2}(\tan\frac{1}{2}(90 - \phi + z) + \tan\frac{1}{2}(90 - \phi - z))$.
It can be shown that
$b^2 - c = (\frac{R}{2}(\tan\frac{1}{2}(90 - \phi + z) - \tan\frac{1}{2}(90 - \phi - z)))^2$.
These correspond to the values for the centre and radius of the almucantar circles as found in the Islamic tradition (al-Farghānī, ca. 850 CE).

4. Historical connections? Gaultier introduced his orthogonal systems of coaxal circles in 1813 in order to solve geometrical problems.

Example: circle tangent to three given circles.
You see circles of the two systems in his Fig. 11.

No relationship to the astrolabe at all.



The “so-what?” question: why is this interesting?

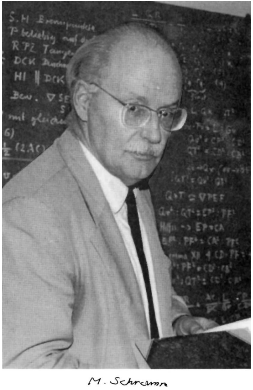
My personal answer: The similarity tells something philosophically interesting about mathematics.

The similarity is between mathematical patterns in different centuries and different cultures.

Mathematics transcends time and place to some extent

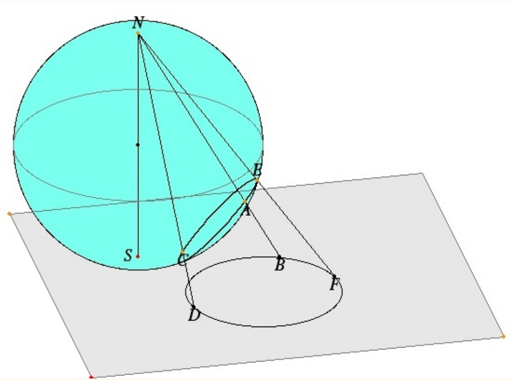
The German philosopher Mattias Schramm said: Mathematics and exact sciences seem to be a witness of the unity of mankind.

Matthias Schramm, 1928–2005

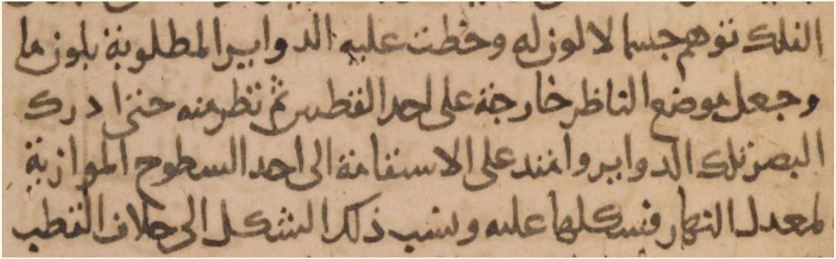


What mathematics did the medieval Islamic astrolabe makers use?

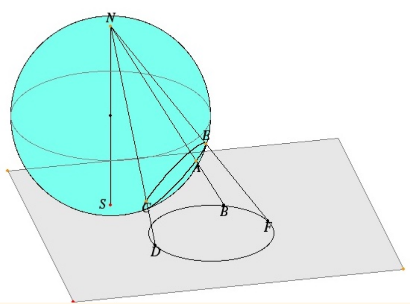
Al-Biruni (Ca. 1000 CE), stereographic projection: imagine the heavens as a sphere without colour, and the circles you want to study are imagined as coloured. The eye sits at one of the poles (here N) and it sees the coloured circles in the plane of the equator or a plane parallel to it.



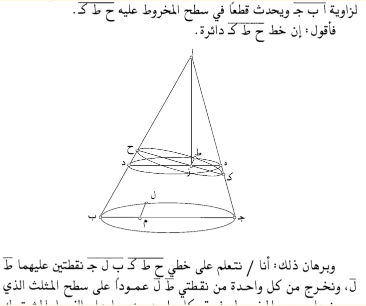
Al-Biruni on the celestial sphere and stereograpic projection in Arabic


الفلك نتوهم جسما لا لون له وخطت عليه الدوائر المطلوبة بلونها
وجعل موضع الناظر خارجه على احد القطبين ثم نظر منه حتى ادرك
البصر تلك الدوائر وامتد على الاستقامة الى احد السطوح الموازية
لمعدل النهار فتشكلها عليه وتشبه ذلك الشكل الى خلاف القطب

They could prove that stereographic projection preserves circles.



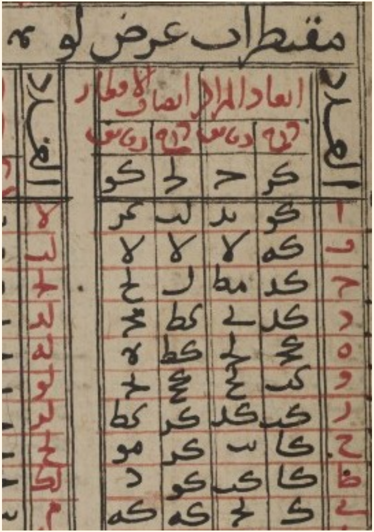
The proof is based on Apollonius of Perga (200 BCE), *Conics* Book 1 prop. 5.

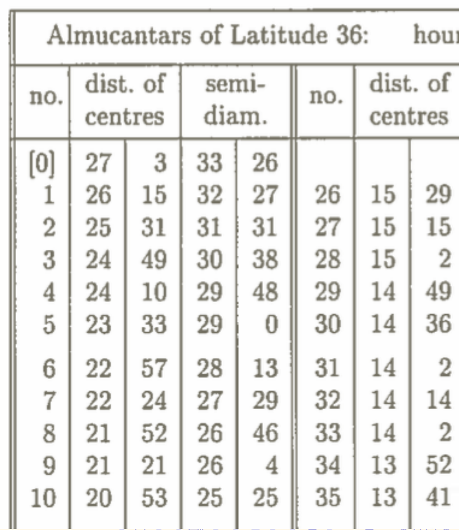


لزاوية ا ب جـ ويحدث قطعاً في سطح المخروط عليه ح ط كـ.
فأقول: إن خط ح ط كـ دائرة.

وبرهان ذلك: أنا / نتعلم على خطي ح ط كـ ب ل جـ نقطتين عليهما ط
ل، ونخرج من كل واحدة من نقطتي ط ل عمـوداً على سطح المثلث الذي

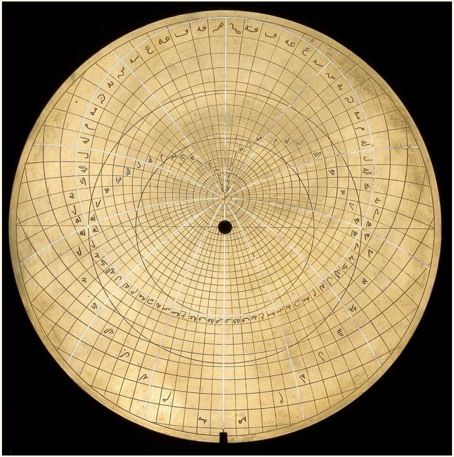
Al-Farghani (ca. 850 CE) computed centres and radii of almucantar circles (based on tangent functions)





Almucantars of Latitude 36: hour								
no.	dist. of centres		semi-diam.		no.	dist. of centres		
[0]	27	3	33	26				
1	26	15	32	27	26	15	29	
2	25	31	31	31	27	15	15	
3	24	49	30	38	28	15	2	
4	24	10	29	48	29	14	49	
5	23	33	29	0	30	14	36	
6	22	57	28	13	31	14	2	
7	22	24	27	29	32	14	14	
8	21	52	26	46	33	14	2	
9	21	21	26	4	34	13	52	
10	20	53	25	25	35	13	41	

A mystery: Did the Islamic astrolabe makers know that the almucantars and azimuthal circles are orthogonal?
If they knew it, could the mathematicians prove it?



Astrolabe by Allahdad, ca. 1570, Lahore
Plate for latitude $66^o30'$ for transformation of equatorial to ecliptical coordinates
Oxford, History of Science Museum, no. 47376
Sarma no. A002.

Thank you for your attention!

Download this presentation on www.jphogendijk.nl/circles.pdf

Literature: S.R. Sarma, *A Descriptive Catalogue of Indian Astronomical Instruments*, available on www.srsarma.in

On astrolabes in general: J.E. Morrison, *The Astrolabe*, available at archive.org

On coaxal circles: see for example Dan Pedoe, *Circles: A Mathematical View*, Washington DC, Mathematical Association of America, 1995.

L. Gaultier, Mémoire sur les moyens généraux de construire graphiquement un Cercle déterminé par trois conditions, et une Sphère déterminée par quatre conditions, *Journal de l'École Polytechnique, Seizième Cahier, Tome IX*, 1813, pp. 124-214. Available online via gallica.fr